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Estimation of a Satellite Lifetime in Orbit

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Abstract—An approximate technique for calculating the satellite orbit decay under the aerodynamic drag effect for arbitrary values of the orbit eccentricity and for the realistic model of the altitude dependence of the atmospheric density is proposed.

Consider the problem of determination of the lifetime of a satellite moving over the near-Earth elliptical orbit. The direct integration of the equations of motion of a satellite, even with ignoring its angular motion, requires large computer time expenses. By this reason, it is a common practice in this case to use one of the forms of the averaged system of equations describing the variation of "osculating" elements of the orbit (see [1]–[4]). Neglecting the rotation of the Earth and the non-central character of the gravity field, we write the equations of motion of a satellite as [2]

$$\frac{d^2u}{d\varphi^2} + u = \eta, \qquad (1)$$

$$\frac{d\eta}{d\varphi} = \frac{\eta C_x SR}{m}\frac{\rho(u)}{u}\sqrt{1+\left(\frac{1}{u}\frac{du}{d\varphi}\right)^2}, \qquad (2)$$

where $u = \frac{R}{r}$; $\eta = \frac{gR^2}{\left(r^2\frac{d\varphi}{dt}\right)^2} = \frac{R}{p}$ is the quantity, which is invariant over the Keplerian trajectory and slowly changes during the satellite deceleration; φ is the geocentric angle; p is the ellipse parameter; r is the distance to the Earth's center; R is the radius of the Earth; g is the gravity acceleration on the Earth's surface; ρ is the atmospheric density; C_x, S, and m are the drag coefficient, the characteristic area and the mass of a satellite. The values of $u_p = \frac{R}{r_p} = \frac{R(1+\varepsilon)}{p}$ and $u_a = \frac{R}{r_a} = \frac{R(1-\varepsilon)}{p}$ correspond to the points of perigee ($r = r_p$) and apogee ($r = r_a$). Here, ε is the eccentricity of the orbit.

We take advantage of one of versions of the averaging method described and substantiated in [5] (see also [6]). We consider the aerodynamic drag of a satellite to be small (low atmospheric density); then η represents a quantity slowly varying during the flight, and the variable $u(\varphi)$ over one period of revolution is described by the formula

$$u = \eta + (u_p - \eta)\cos\varphi. \qquad (3)$$

We define the integral of action, corresponding to equation (1), at the "frozen" value of η as

$$J = \int_0^{2\pi}\left(\frac{du}{d\varphi}\right)^2 d\varphi = 2\int_{2\eta-u_p}^{u_p}\frac{du}{d\varphi}(u)\,du = \pi(u_p-\eta)^2. \qquad (4)$$

Here, the integral is taken over one period of satellite's revolution around the Earth, and the function $\frac{du}{d\varphi}(u)$ on a growing branch of $u(\varphi)$ is determined by the following formula

$$\frac{du}{d\varphi}(u) = \sqrt{(u_p-\eta)^2-(u-\eta)^2}. \qquad (5)$$

According to [5], the equations describing the variation of osculating elements $u_p = \frac{R}{r_p}$ and variation of the period-averaged value $\eta = \frac{R}{p}$, for which we retain the former designation, have the form:

$$\frac{\partial J}{\partial u_p}\frac{du_p}{d\varphi} + \frac{\partial J}{\partial\eta}\overline{\overline{\frac{d\eta}{d\varphi}}} = 0, \qquad (6)$$

$$\frac{d\eta}{d\varphi} = \overline{\frac{d\eta}{d\varphi}}, \qquad (7)$$

where $\overline{\frac{d\eta}{d\varphi}}$ and $\overline{\overline{\frac{d\eta}{d\varphi}}}$ denote the right-hand side of relation (2), averaged over the period of revolution, with a weight of 1 and $\frac{\partial}{\partial\eta}\left\{\left[\frac{du}{d\varphi}(u)\right]^2\right\}$, respectively.

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Taking into account that

$$\frac{\partial J}{\partial \eta} = \frac{\partial J}{\partial u_p} = 2\pi(u_p - \eta), \tag{8}$$

we obtain the system of equations

$$\frac{d\eta}{d\varphi} = \frac{C_x SR\eta}{m} \frac{\int\limits_{2\eta - u_p}^{u_p} \frac{\rho(u)}{u} \sqrt{1 + \left[\frac{1}{u}\frac{du}{d\varphi}(u)\right]^2} \frac{du}{\frac{du}{d\varphi}(u)}}{\pi}, \tag{9}$$

$$\frac{du_p}{d\varphi} = \overline{\overline{\frac{d\eta}{d\varphi}}} = \frac{C_x SR\eta}{m}$$

$$\times \frac{\int\limits_{2\eta - u_p}^{u_p} \frac{\rho(u)}{u} \sqrt{1 + \left[\frac{1}{u}\frac{du}{d\varphi}(u)\right]^2} \frac{\partial}{\partial \eta}\left[\frac{du}{d\varphi}(u)\right] du}{\tilde{J}}, \tag{10}$$

where, in accordance with [5],

$$\frac{\partial}{\partial \eta}\left[\frac{du}{d\varphi}(u)\right] = \frac{u_p - u}{\sqrt{(u_p - \eta)^2 - (u - \eta)^2}}, \tag{11}$$

and the integral in the denominator of the right-hand side of (10) is easily calculable:

$$\tilde{J} = \int\limits_{2\eta - u_p}^{u_p} \frac{\partial}{\partial \eta}\left[\frac{du}{d\varphi}(u)\right] du = \pi(u_p - \eta). \tag{12}$$

We first write out the approximate expressions for integrals in the numerator of the right-hand sides of equations (9) and (10) for the case (a) of low eccentricity, where the following inequality is satisfied:

$$\varepsilon \ll 1, \tag{13}$$

and, hence,

$$\left[\frac{1}{u}\frac{du}{d\varphi}(u)\right]^2 \ll 1. \tag{14}$$

We suppose that for $r \geq r_p$ the density is an exponential function of the altitude

$$\rho(r) = \rho_p e^{-\lambda(r - r_p)}, \tag{15}$$

where $\rho_p = \rho(r_p)$.

In the case under consideration, when calculating the integrals in the numerator of expressions (9) and (10) it is convenient to pass to the independent variable φ and to obtain the well-known result (see, e.g., [3]):

$$\frac{dp}{d\varphi} = -\frac{R}{\eta^2}\frac{d\eta}{d\varphi} \approx -\frac{C_x S p^2 \rho_p}{2\pi m} \int\limits_0^{2\pi} e^{-\frac{\lambda p \varepsilon}{1+\varepsilon}(1 - \cos\varphi)} d\varphi$$

$$= -\frac{C_x S p^2 \rho_p}{m} e^{-\lambda(p - r_p)} \int\limits_0^{2\pi} e^{\lambda(p - r_p)\cos\varphi} d\varphi \tag{16}$$

$$= -\frac{C_x S p^2 \rho_p}{2\pi m} e^{-x} I_0(x),$$

where $x = \lambda(p - r_p)$, $I_0(x)$ is the modified Bessel function (see [7]).

Similarly, we obtain

$$\frac{dr_p}{d\varphi} = -\frac{R}{u_p^2}\frac{du_p}{d\varphi} = -\frac{C_x S p^2 \rho_p}{m} e^{-x}[I_0(x) - I_1(x)]. \tag{17}$$

In deriving formulas (16) and (17) we have used the relation ([7]):

$$\frac{1}{2\pi}\int\limits_0^{2\pi} e^{x\cos\varphi} \cos n\varphi d\varphi = I_n(x). \tag{18}$$

Note that the formulas written above are based only on assumption (13), but the parameter $\lambda p \varepsilon$ is not considered to be small, since the value of λp for the Earth equals 100–1000 (remind that in the analysis of reentry trajectories the inequality $R\lambda \gg 1$ is frequently used, [8]).

Consider now the other limiting case (b), where the osculating ellipse is strongly elongated; in this case $\varepsilon = O(1)$,

$$\rho(r_a) \ll \rho(r_p). \tag{19}$$

In this case, calculating the integrals in the numerator of right-hand sides of (9) and (10) it is convenient to use the altitude as the variable of integration and to replace the upper limit of integration by the infinity, taking (19) into account.

Assuming the value of λp to be a “large” parameter, we obtain the asymptotic estimate:

$$\frac{dp}{d\varphi} = -\frac{R}{\eta^2}\frac{d\eta}{d\varphi}$$

$$\approx -\frac{C_x S p}{\pi m}\int\limits_{r_p}^{r_a} \rho(r) \sqrt{1 + \frac{p r_p}{(r - r_p)\left[p\left(1 + \frac{r}{r_p}\right) - 2r_p\right]}} dr \tag{20}$$

$$\approx -\frac{C_x S p \rho_p}{\pi m}\sqrt{\frac{p r_p}{2(p - r_p)}}\int\limits_{r_p}^{\infty} \frac{e^{-\lambda(r - r_p)} dr}{\sqrt{(r - r_p)[1 + O(r - r_p)]}}$$